Computing FSS Earth Station Antenna Gain in the Direction of a Fixed Terrestrial Station

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Note: This paper was prepared as a technical contribution for consideration by the WinnForum Spectrum Sharing Committee in support of industry standards development for the 3.5 GHz Citizens Broadband Radio Service. This paper is not an FCC proposal or requirement. The views expressed in this paper are those of the authors and may not necessarily represent the views of the Federal Communications Commission.

Background and Overview

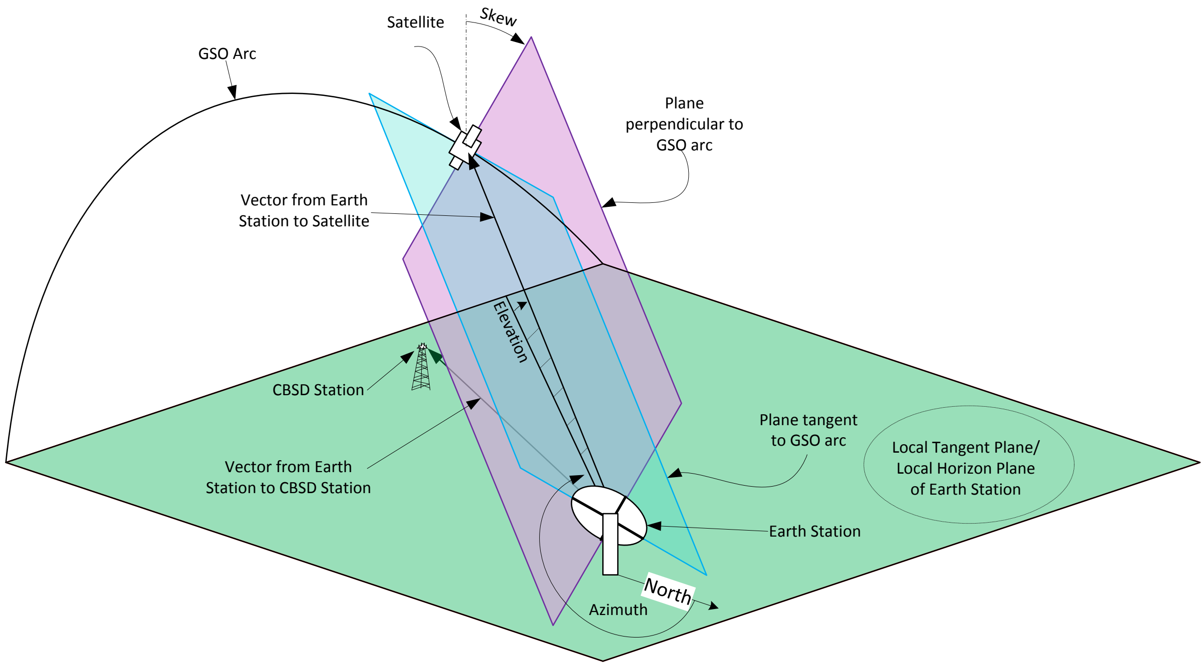
Spectrum management in frequency bands shared between fixed satellite service (FSS) earth stations and fixed terrestrial radios may require the calculation of terrestrial emitter RF power at the input of an earth station receiver to ensure the earth station interference threshold is not exceeded. The potential received interference power from terrestrial emitters will depend, in part, on the gain of the FSS earth station antenna in the direction of the terrestrial emitter. In general, §25.209 of the Commission's rules specifies maximum earth station antenna gain for several frequency bands, however the rules do not specify how to calculate the antenna gain limits in a specific direction from the earth station. This document presents three methods for consideration in computing earth station antenna gain limits that are subject to and derived from §25.209 limits. The calculations in this document focus on the spectrum sharing requirements in the Part 96 rules for the 3.5 GHz band, and the methods should be generally applicable to other frequency bands shared between FSS earth stations and fixed terrestrial radios.

Spectrum sharing in the 3.5 GHz band requires commercial users (priority access and general authorized access) under Part 96 to protect existing FSS earth stations as stated in §96.17. The calculations of interference power from terrestrial Citizens Band Service Devices (CBSDs) to an FSS earth station requires the use of a reference antenna system specified in §96.17. The antenna gain limits of the earth station reference antenna are specified in 25.209(a)(1) and 25.209(a)(4) of the Commission's rules as shown and stated in Figure 1. Given a prospective location of a CBSD, the location of the earth station, and the GSO satellite to which the earth station antenna is pointing, Spectrum Access System (SAS)[1] operators will be required to calculate the aggregate interference power at the input of the earth station receiver, to ensure that the aggregate interference threshold is not exceeded.

While the Commission's Part 96 rules specify the earth station interference threshold and the reference antenna system, the rules do not specify a method for calculating the earth station antenna gain for CBSD positions at angles that are not in the perpendicular or tangent planes. At small angular offsets from the earth station antenna boresight, CBSD transmitters will have a much greater interference effect (high earth station antenna gain) compared to CBSD transmitters at high angular offsets from earth station antenna boresight direction (low antenna gain). The relative power difference can be as much 1,000 times or more (> 30 dB), so it is important that SAS operators perform interference calculations using consistent calculations of earth station antenna gain.

Alternative industry approaches to these calculations may be applied, and this document offers three methods for consideration, given the location of a terrestrial radio, the location of an FSS earth station and the GSO satellite position to which the earth station antenna is pointing. Comparing the results of different methods offers the value of a) cross-checking the results of each implementation for a common set of inputs, to ensure that the calculations are correct, b) describing different mathematical approaches where different readers may find one approach to be more understandable than others, and c) presenting alternative formulas and calculation methods for industry consideration in SAS operational software development and testing. Each of the methods are closed form calculations that do not require iterative methods (e.g., Vincenty method).[2]

[1] SAS is a system that authorizes and manages use of spectrum for the Citizens Broadband Radio Service.
[2] See, https://webgis.wr.usgs.gov/pigwad/docs/Geodesic_Tools_Manual.pdf



§25.209 Earth station antenna performance standards

(a) Except as provided in paragraph (f) of this section, the co-polarization gain of any earth station antenna operating in the FSS and transmitting to a GSO satellite, including earth stations providing feeder links for satellite services other than FSS, may not exceed the following limits:

(1) In the plane tangent to the GSO arc, as defined in §25.103, for earth stations not operating in the conventional Ku-band, the 24.75-25.25 GHz band, or the 28.35-30 GHz band:

$29 - 25\log_{10}\theta$	dBi	for	$1.5° <= \theta <= 7°$
8	dBi	for	$7° < \theta <= 9.2°$
$32 - 25\log_{10}\theta$	dBi	for	$9.2° < \theta <= 48°$
-10	dBi	for	$48° < \theta <= 180°$

where θ is the angle in degrees from a line from the earth station antenna to the assigned orbital location of the target satellite, and dBi refers to dB relative to an isotropic radiator.

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(4) In the plane perpendicular to the GSO arc, as defined in 25.103, for earth stations not operating in the conventional Ku-band, the 24.75-25.25 GHz band, or the 28.35-30 GHz band:

$32 - 25\log_{10}\theta$	dBi	for	$3° < \theta <= 48°$
-10	dBi	for	$48° < \theta <= 180°$

where θ and dBi are as defined in paragraph (a)(1) of this section.

Figure 1 – Earth station antenna gain limits in GSO arc planes

Earth Station Antenna Gain Calculation Overview

The following overview highlights the calculation logic of the three methods, where the nature and sequence of the overview description follows the sequence of calculations presented in Method 1. The three methods produce the same results but are mathematically performed differently. The commonality and differences between the three methods include such issues as; the use of different types of coordinate systems (earth-centered Cartesian, spherical, polar, etc.), the manner of translating vectors between coordinate systems, the use (or not) of vector and/or coordinate frame rotations, the use (or not) of vector dot products, and the use (or not) of commonly used trigonometric formulas used in satellite engineering and spherical earth navigation and surveillance analysis. All three of the methods assume that the FSS gain limits at angles between the GSO arc perpendicular and tangent planes form an ellipse, with vertices from the tables in §25.209(a)(1) and §25.209(a)(4). All three methods account for the polarization angle (skew) of the FSS antenna.[3] Parameter labels (and coordinate system naming) have not been unified between the three method descriptions.

In summary, the problem, inputs, and calculation are as follows:

Problem:

Calculate the FSS antenna gain limit toward a CBSD

- Per §96.17 (Protection of Existing FSS Earth Stations), which references,
- §25.209(a)(1) [plane tangent to GSO arc] and §25.209(a)(4) [plane perpendicular to GSO arc]

Inputs:

Location of

- FSS earth station (Latitude, Longitude, Height above sea level)
- CBSD (Latitude, Longitude, Height above sea level)
- Satellite (GSO Longitude)

Calculate:

Angle θ between

- Boresight of FSS earth station antenna (toward GSO satellite), and,
- Vector from earth station to CBSD

Determine the gain limit in the GSO arc perpendicular plane

- Using the angle above, by
- Using the formula in §25.209(a)(4)

If the angle θ is > 9.2°,

- The calculated gain limit from either §25.209(a)(1) or §25.209(a)(4) answers the problem.

[3] See *Satellite Communications Systems*; Fifth edition, G. Maral and M. Bosquet, 2009 Wiley and Sons; Section 8.3.5, Pointing angles of an earth station antenna, Sec. 8.3.5.2, for definition of polarization angle (skew). See Figure 2.

- i.e., at angles $\theta > 9.2°$, the earth station antenna gain limit is concentric around the boresight (circular).

If the angle θ is <= 9.2°, the gain limits in §25.209(a)(1) and §25.209(a)(4) may differ by up to 3 dB; then,

- Assume the gain limit at angles between the planes perpendicular and tangent to the GSO arc forms an ellipse,
- With vertices A and B:
  - A = Gain limit from §25.209(a)(1), and
  - B = Gain limit from §25.209(a)(4)
- The gain limit toward the CBSD is calculated based on:
  - The angle of the CBSD vector between the two planes, which requires calculation of,
    - The polarization angle (or skew) of the GSO arc planes from the local vertical & horizon

Software Implementation of Each Calculation Method and Test Matrix[4]

Excel spreadsheet is attached to this PDF file

Excel Sheets

"Studies" sheet	[Input and output data of all three methods; comparison of results]
"Method 1 calculations" sheet	[Method 1 input and output computation data]
"Method 2 calculations" sheet	[Method 2 input and output computation data]
"Method 3 calculations" sheet	[Method 3 input and output computation data]
"Create Matrix for Studies" sheet	[Input parameters for creating a matrix of terrestrial stations around an FSS earth station]

Visual Basic Modules

Calculate	[Visual Basic for Applications (VBA) control procedure that creates a matrix of multiple terrestrial stations from parameters in the "Create Matrix for Studies" sheet; sets the input parameters for Methods 1 and 2, and executes the Method 3 VBA routines in Module 3; copies results of all three methods to "Studies" sheet]
Module 3	[VBA module with Method 3 computations]

Macros

ComputeESGain	[Method 3 calculation procedures in "Module 3"]
IterateStations	[The VBA control 'sub' procedure in the "Calculate" module]

[4] Note: For a given earth station and a given satellite, the azimuth, elevation, and skew only need to be computed once. For simplicity of implementation here, these computations are repeated for each CBSD location.

Quick Start Guide to Excel / VBA Calculations

Each calculation method (Method 1, Method 2, and Method 3) can be executed individually, or all three executed simultaneously using an input matrix of multiple CBSD locations.[5] Input fields are highlighted in yellow, and outputs are highlighted in green.

- **Individually**, given the location of an earth station, GSO satellite longitude, and CBSD location;
    - User inputs the location parameters highlighted in the yellow fields:
        - Method 1 => "Method 1 calculations" sheet
        - Method 2 => "Method 2 calculations" sheet
        - Method 3 => "Method 3 calculations" sheet
    - Calculated results are shown in the highlighted green fields in each respective Excel sheet
        - Method 1 and 2 routines are written in Excel, and they execute immediately when the input parameters are entered, with the results shown in the green fields.
        - Method 3 is a Visual Basic routine that runs when the "ComputeESGain" macro routine is executed, with results shown in the "Method 3 calculations" sheet in green.
            - To execute Method 3 calculations:
                - Enter input parameters (yellow) in the "Method 3 data" sheet
                - Click the "Run Method 3" button in the "Method 3 data" sheet
                - Calculated results are shown in green
- (or, all three methods executed) **Simultaneously**, given a matrix of multiple CBSD locations (created in the "Studies" sheet), for a given earth station and GSO longitude. The matrix is created by the VBA macro "IterateStations", using relative azimuth, distance, and height parameters in the "Create Matrix for Studies" sheet. The calculated results of all three methods are written in the "Studies" sheet, along with the input parameters, for each CBSD location.
    - User enters location parameters in the "Create Matrix for Studies" sheet (yellow fields)
        - Earth station location and GSO satellite longitude (left side of the sheet)
        - Multiple CBSD location parameters (right side of the sheet)
            - Azimuth angles relative to the earth station pointing azimuth (Col. M)
            - Vector of distances from the earth station, for each azimuth angle (Col. N)
            - Vector of heights, above / below earth station height, for each azimuth angle and distance (Col. O)
            - Note: The number of parameters in each dimension is variable; the first blank entry in each column denotes the end of the parameter list in that column
    - Click the "Run Studies" button in the "Create Matrix for Studies" sheet
        - Calculated results are shown in the "Studies" sheet, along with the calculated differences in polarization and gain between each method.

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[5] Excel Macros should be enabled (File -> Options -> Trust Center -> Trust Center Settings -> Macro Settings -> (check) Enable all macros -> OK.

METHOD 1

Outline of calculation method:

This method uses textbook formulas for calculating the azimuth / elevation angles and vectors toward the satellite, and the CBSD, in the local earth station coordinate system. This is followed by a rotation of the local earth station axes based on the satellite skew angle, to determine the direction of the CBSD and the gain of the earth station, relative to the GSO arc perpendicular and tangent planes around the earth station boresight.

A. Calculate the angle (EC)[6] between the boresight of the **E**arth station antenna and a vector in the direction of the **C**BSD, using a local horizon plane reference.[7] If that angle is greater than 9.2°, the gain limit toward the CBSD is the value taken from either §25.209(a)(1) or §25.209(a)(4) for that angle. For angles greater than 9.2°, the earth station antenna gain limit is concentric around the earth station antenna boresight (circular); that is, the gain is independent of the rotation of the polarization plane around the antenna boresight (the antenna boresight axis is common to both the GSO arch tangent plane and the GSO arc perpendicular plane).

B. This step is needed If the angle (EC) between the earth station boresight and the CBSD is less than or equal to 9.2°; the orientation of the GSO arc reference planes need to be accounted for in the calculation, because there can be up to a 3 dB difference in the earth station gain limit for a small angle in the perpendicular plane compared to the same angle in the tangent plane. For EC <= 9.2°:
    1. Rotate the axes of the earth centered local horizon plane to align with the earth station boresight and the tangent plane of the GSO arc. This new frame of reference is earth station antenna centered, with the X' axis toward the satellite, X'-Y' plane tangent to the GSO arc, and X'-Z' plane perpendicular to the GSO arc.
    2. Project the vector **C** (toward the CBSD) into this new reference frame and calculate the earth station antenna gain limit toward the CBSD from its angular offset between the two GSO arc planes.[8]

[6] In Figure 1, EC is the angle between the lines labeled "Vector from Earth Station to Satellite" and "Vector from Earth Station to CBSD Station".

[7] At the FSS earth station antenna; X-axis toward true North, Y-axis East, Z-axis toward the center of the earth (spherical earth, right handed coordinate system).

[8] Assume the antenna gain for a given angle EC forms an ellipse around the boresight axis. Figure 5 shows a diagram of the ellipse and the gain values in the Y'-Z' plane, where the vertices of the ellipse are the values $G(@EC)_{§25.209(a)(1)}$ on the Y' axis in the GSO tangent plane, and $G(@EC)_{§25.209(a)(4)}$ on the Z' axis in the GSO perpendicular plane. The calculated gain toward the CBSD will be between these two values, depending on the angular offset between the GSO arc planes, which is the angular rotation of these planes around the earth station boresight axis.

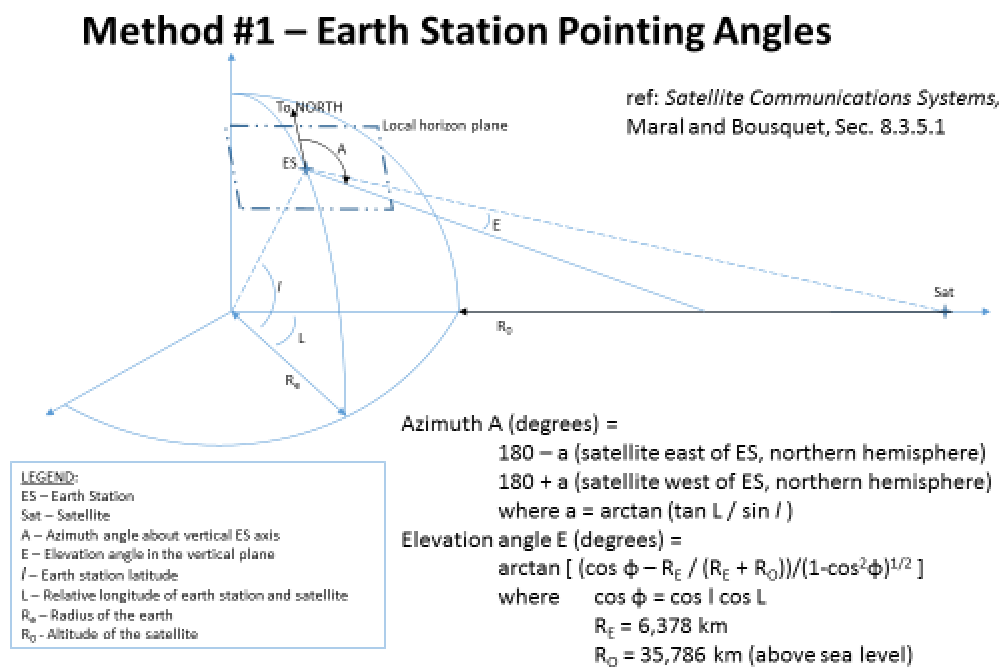

Figure 2 – Method #1 – Earth Station Pointing Angles

Calculation algorithm and formulas in more detail:

A. Calculate the angle (EC) between the earth station boresight and the CBSD direction, and calculate the earth station gain toward CBSD from §25.209(a)(1) and §25.209(a)(4). If EC > 9.2°, the gain in both planes are the same and this gain equals the gain, $G_{C/E}$, toward the CBSD.
   a. Calculate the FSS earth station azimuth and elevation angle (in the local horizon plane) toward the GSO satellite; “A” and “E” in Figure 2.[9]
      i. Azimuth A (degrees) = $E_{GSO\text{-}Az}$[10]
         - 180° – a (satellite east of earth station)
         - 180° + a (satellite west of earth station)
         - a = arctan(tan L / sin l)
            - L is the absolute value of the difference between the longitude of the earth station and longitude of the satellite
            - l is the latitude of the earth station
      ii. Elevation angle E (degrees) = $E_{GSO\text{-}El}$
         - $E_{GSO\text{-}El}$ = arctan[(cos φ – R_E / (R_E + R_O))/(1-cos$^2\varphi$)$^{1/2}$]
         - cos φ = cos l cos L
         - R_E = Earth radius = 6,378 km

[9] *Satellite Communications Systems*; Fifth edition, G. Maral and M. Bosquet; Section 8.3.5, Pointing angles of an earth station antenna (see Figure 2); Note, the vector **E** (“Vector from Earth Station to Satellite”), and components $E_{GSO\text{-}Az}$ and $E_{GSO\text{-}El}$, are terms used in the Excel “Method 1 calculations” sheet.
[10] In the northern hemisphere.

- R_O = Satellite altitude (above sea level) = 35,786 km

iii. Vector **E**(x,y,z) = **E**(cos (E_{GSO-Az}), sin (E_{GSO-Az}), - sin (E_{GSO-El})); earth station boresight toward GSO satellite (see "Method 1 Calculations" Excel sheet).

b. Calculate the azimuth and elevation angle in the direction of the CBSD antenna (at location "S") from the FSS earth station antenna (at location "U"). This is vector **C** ("Vector from Earth Station to CBSD Station") in the local horizon plane. From Geyer, Section 4.2, The Indirect Problem of Geodesy,[11]

  i. Azimuth (C_{Az})[12] =
    - arctan { $\cos(L_S)\sin(\lambda_S - \lambda_U)/[\sin(L_S)\cos(L_U) - \cos(L_S)\sin(L_U)\cos(\lambda_S - \lambda_U)]$ }
    - L is the latitude of the CBSD antenna at location "S" and the earth station antenna at location "U"[13]
    - λ is the longitude of the CBSD antenna at location "S" and the earth station antenna at location "U"

  ii. Elevation angle (C_{El}) = the inverse cosine (minus 90°) of the angle between the line from the FSS earth station to the CBSD and the line from the earth station to the center of the earth. This follows from the law of cosines; see Figure 3.
  C_{El} =
    - arccos { $(D_{C/E}^2 + C^2 - B^2) / 2 D_{C/E} C$} - 90°
      - $D_{C/E}$ – Distance between CBSD "C" at location "S" and FSS earth station "E" at location "U"
        - $D_{C/E}$ = Geocentric angle θ (radians) between "S" and "U" multiplied by the radius of the earth
        - $D_{C/E} = (\theta_{rad}) \times R_E$
        - R_E = Earth radius = 6,378 km
        - θ = arccos { $\cos(L_U)\cos(L_S)\cos(\lambda_U-\lambda_S)+\sin(L_U)\sin(L_S)$ }[14]
      - $C = R_E + H_E$
        - H_E – Mean sea level height of the FSS earth station antenna
      - $B = R_E + H_C$
        - H_C – Mean sea level height of the CBSD antenna

  iii. Vector **C**(x,y,z) = **C**(cos (C_{Az}), sin (C_{Az}), - sin (C_{El})); vector from earth station toward the CBSD (see "Method 1 Calculations" Excel sheet).

c. Calculate the angle, EC, between vectors **E** and **C**.

  i. EC=arccos (**E** · **C** / {||**E**|| * ||**C**||})[15]
    - EC=arccos { $[(E_x)(C_x)+(E_y)(C_y)+(E_z)(C_z)] / [(E_x^2+E_y^2+E_z^2)^{1/2} \times (C_x^2+C_y^2+C_z^2)^{1/2}]$ }
    - EC=arccos {

[11] Aircraft Navigation and Surveillance Analysis for a Spherical Earth, M. Geyer, October 2014, DOT-VNTSC-FAA-15-01, Federal Aviation Administration; Section 4.2, Indirect Problem of Geodesy.
[12] Equation 70, page 43, Section 4.2.2 Computing the Azimuth Angles of the Connecting Arc, DOT-VNTSC-FAA-15-01
[13] The letter "L" in Maral and Bousquet is the difference in longitude, whereas in Geyer, the letter "L" denotes latitude.
[14] Equation 62, page 41, Section 4.2.2 Computing the Azimuth Angles of the Connecting Arc, DOT-VNTSC-FAA-15-01
[15] From the definition of the dot product of two Euclidean vectors, the cosine of the angle between two vectors equals the ratio of the dot product of the two vectors divided by the magnitudes of the two vectors; **a** · **b** = ||**a**|| x ||**b**|| cos θ, where θ is the angle between vectors **a** and **b**, and ||**a**|| is the magnitude of vector **a** and ||**b**|| is the magnitude of vector **b**.

$(\cos(E_{GSO\text{-}Az})\cos(C_{Az}) + \sin(E_{GSO\text{-}Az})\sin(C_{Az}) + \sin(E_{GSO\text{-}El})\sin(C_{El}))$ /

$(\cos^2(E_{GSO\text{-}Az}) + \sin^2(E_{GSO\text{-}Az}) + \sin^2(E_{GSO\text{-}El}))^{1/2}$ /

$(\cos^2(C_{Az}) + \sin^2(C_{Az}) + \sin^2(C_{El}))^{1/2}$

}

ii. Use the tables in §25.209(a) to determine the FSS antenna gain limit in the GSO arc tangent plane (§25.209(a)(1)) and the GSO arc perpendicular plane (§25.209(a)(4)). If EC is greater than 9.2°, the gain limits in both planes are the same, and the value from either table is the gain toward the CBSD, $G_{C/E}$, at angle EC from the FSS earth station antenna boresight. The computation is DONE. If EC is less than or equal to 9.2°, step B below is performed to calculate the gain limit, based on the polarization angle (or skew) of the GSO arc tangent and perpendicular planes around the axis of the FSS antenna boresight. As shown in Maral and Bousquet, Section 8.3.5.2 Polarization angle, the polarization angle is a function of the earth station latitude and the difference in longitude of the earth station relative to that of the satellite. See Figures 2 and 4.

B. For CBSD angles to the earth station boresight (EC) less than or equal to 9.2°, calculate the FSS earth station antenna gain limit toward the CBSD, $G_{C/E}$, based on the GSO tangent plane and perpendicular plane gain limits determined immediately above (A.c.ii). In general, for angular offsets (EC) <= 9.2°, the gain toward the CBSD will be a value between the gain limit in the perpendicular plane and the gain limit in the tangent plane. The gain value between these two limits will depend on the angular rotational position of vector **C,** relative to the two GSO arc planes around the boresight axis of the earth station.

Vector **C** (toward the CBSD) and vector **E** (the earth station boresight axis) were calculated in Step A above, within a local horizon {x, y, z} coordinate system where the X-axis points toward true North, Y-axis East, and the Z-axis toward the center of the earth (spherical earth, right handed coordinate system). There are two unknowns, 1) the angular offset of vector **C** relative to the GSO arc planes, after accounting for the rotational skew of the GSO arc planes around the earth station boresight axis (this axis is vector **E**), and 2) the variation in earth station gain limits at rotational angles between the two GSO arc planes. For a given absolute angle between **C** and **E**, solving these two unknowns will solve the question of the gain limit toward the CBSD, $G_{C/E}$.

The axes of the local horizon plane are rotated to form a new coordinate system that includes the GSO arc tangent and perpendicular planes (Step 1 below). Within this new coordinate system, the variation in earth station gain limits between the two GSO arc planes is used to calculate the gain limit toward the CBSD (Step 2 below). In this second step, the gain limits in the GSO arc tangent plane and perpendicular plane are shown in §25.209(a)(1) and §25.209(a)(4), respectively. For rotational positions between these two planes, the gain limit can be assumed to scribe an ellipse relative to the two GSO arc planes. This is shown in Figure 5, where the y'-axis is in the tangent plane and the z'-axis is in the perpendicular plane, and the vertices, A and B, are the gain values derived from §25.209(a)(1) and §25.209(a)(4). A = $G_{(a)(1)}(\theta\text{=EC})$ is the gain in the tangent plane in §25.209(a)(1) for θ=EC, and B = $G_{(a)(4)}(\theta\text{=EC})$ is the gain in the perpendicular plane in §25.209(a)(4) for θ=EC. For angles between the y'-axis and z'-axis in Figure 5, the gain will be the length of a vector from the origin to the point on the ellipse. Performing this calculation requires knowledge of the components of vector **C** in a coordinate system that includes the two GSO arc planes.

Step 1: rotate the axes of the earth centered local horizon plane to align with the earth station boresight and the tangent plane of the GSO arc. This is done by:

1. Rotating the local horizon {x, y, z} axes in three dimensions to coincide with the earth station boresight axis (x'), the GSO arc tangent plane (x'-y') and the GSO arc perpendicular plane (x'-z').
   a. The x-axis is rotated clockwise around the z-axis by the "azimuth" of the GSO satellite from the earth station ($E_{GSO\text{-}Az}$)[16]
   b. This axis is then rotated clockwise around the y-axis by the "elevation" angle of the GSO satellite from the earth station ($E_{GSO\text{-}El}$)[17]
   c. This axis is then rotated clockwise around itself (the x-axis) by the "skew", or polarization angle of the GSO satellite from the earth station. The polarization angle is given in Maral / Bousquet, Section 8.3.5.2, Polarization angle, Eq. 8.22 c:
      - $\Psi = \arctan(\sin L / \tan l)$
         - L is the difference between the longitude of the earth station and longitude of the satellite
         - l is the latitude of the earth station
   d. The three rotations can be depicted in one rotation matrix, calculated from the three axes rotations using matrix multiplication
      - $R' = R_X(\Psi)\, R_Y(\omega)\, R_Z(\mu)$ [18]
      - Where: $\omega = E_{GSO\text{-}El}$ and $\mu = E_{GSO\text{-}Az}$

$$\mathrm{Rx}(\Psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Psi & \sin\Psi \\ 0 & -\sin\Psi & \cos\Psi \end{pmatrix}$$

$$\mathrm{Ry}(\omega) = \begin{pmatrix} \cos\omega & 0 & -\sin\omega \\ 0 & 1 & 0 \\ \sin\omega & 0 & \cos\omega \end{pmatrix}$$

$$\mathrm{Rz}(\mu) = \begin{pmatrix} \cos\mu & \sin\mu & 0 \\ -\sin\mu & \cos\mu & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 2: project the vector **C** toward the CBSD into the new reference frame to calculate the earth station antenna gain limit toward the CBSD.

1. **C**' = **R**' **C**

[16] Calculated in A.a.i.
[17] Calculate in A.a.ii.
[18] https://en.wikipedia.org/wiki/Rotation_matrix

2. The angle EC was calculated in Step A, and this angle determines the gain values in the GSO arc planes, which are the vertices of an ellipse in Figure 5. The gain limit toward the CBSD will be the length of a vector in Figure 5 with the same angular rotation as vector **C'** (toward the CBSD). The projection of vector **C** in the new earth-station-antenna-centered {x', y', z'} coordinate system will show the rotational position of vector **C** relative to the GSO arc planes, by the values $C'_{y'}$ and $C'_{z'}$. The rotational angle Δ in Figure 5 is the arctangent of $C'_{y'}/ C'_{z'}$.
   a. $\Delta = \arctan (C'_{y'}/ C'_{z'})$
3. The equation of an ellipse, with vertices $A = G_{(a)(1)}(\theta{=}EC)$ and $B = G_{(a)(4)}(\theta{=}EC)$, is used to calculate the gain, $G_{\mathbf{C/E}}(\Delta)$, toward the CBSD. The equation of an ellipse is $(y'/A)^2 + (z'/B)^2 = 1$; or equivalently, $z'(\Delta) = B \cos(\Delta)$ and $y'(\Delta) = A \sin(\Delta)$.[19] Figure 5 shows the gain in polar coordinates with the angle Δ measured from the major axis. The ellipse's equation is $G_{\mathbf{C/E}}(\Delta) = A * B / [A^2\sin^2(\Delta) + B^2\cos^2(\Delta)]^{1/2}$.
   a. $A = G_{(a)(1)}(\theta{=}EC)$
   b. $B = G_{(a)(4)}(\theta{=}EC)$
   c. $G_{\mathbf{C/E}}(\Delta) = A * B / [A^2\sin^2(\Delta) + B^2\cos^2(\Delta)]^{1/2}$

[19] See https://en.wikipedia.org/wiki/Ellipse

Spherical Geometry – Law of Cosines

$A^2 = B^2 + C^2 - 2BC \cos \alpha$ (A = distance between FSS earth station and CBSD)

$B^2 = A^2 + C^2 - 2AC \cos \beta$ (β = elevation angle of CBSD from FSS earth station + 90°)

R_E – Earth radius

H_E – Earth station height above sea level

H_C – CBSD height above sea level

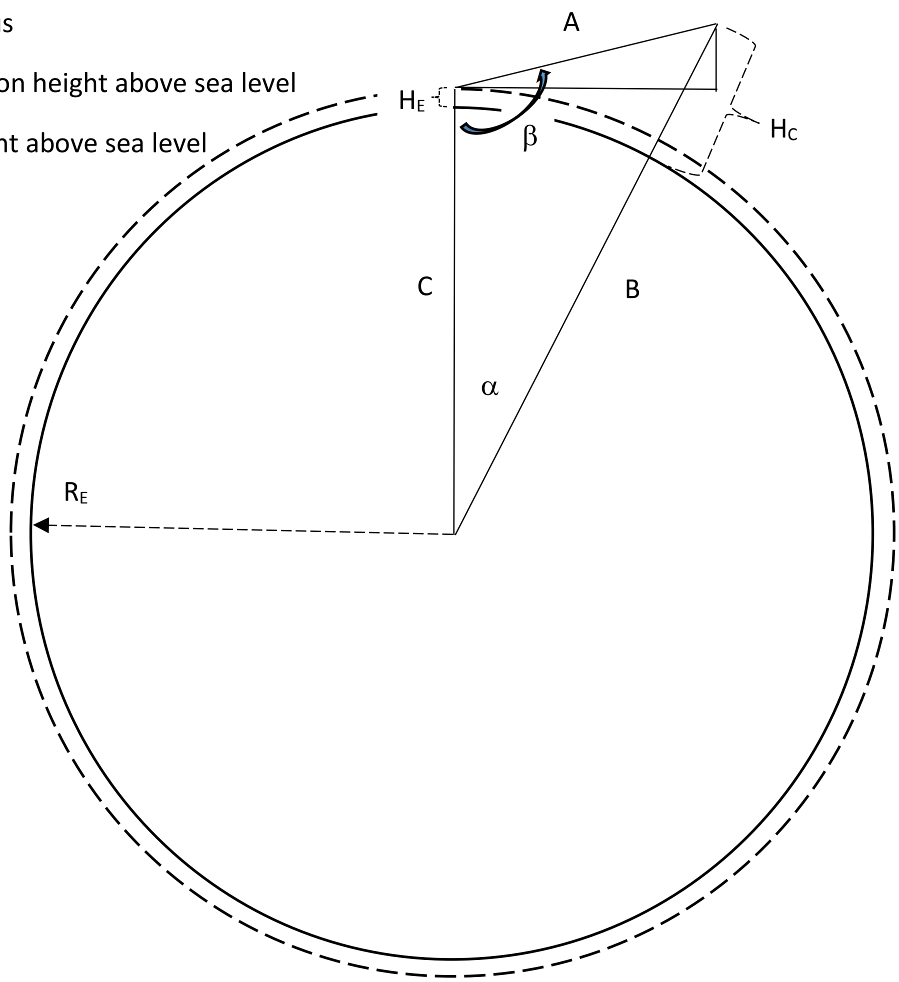

Figure 3 – Spherical Geometry – Law of Cosines

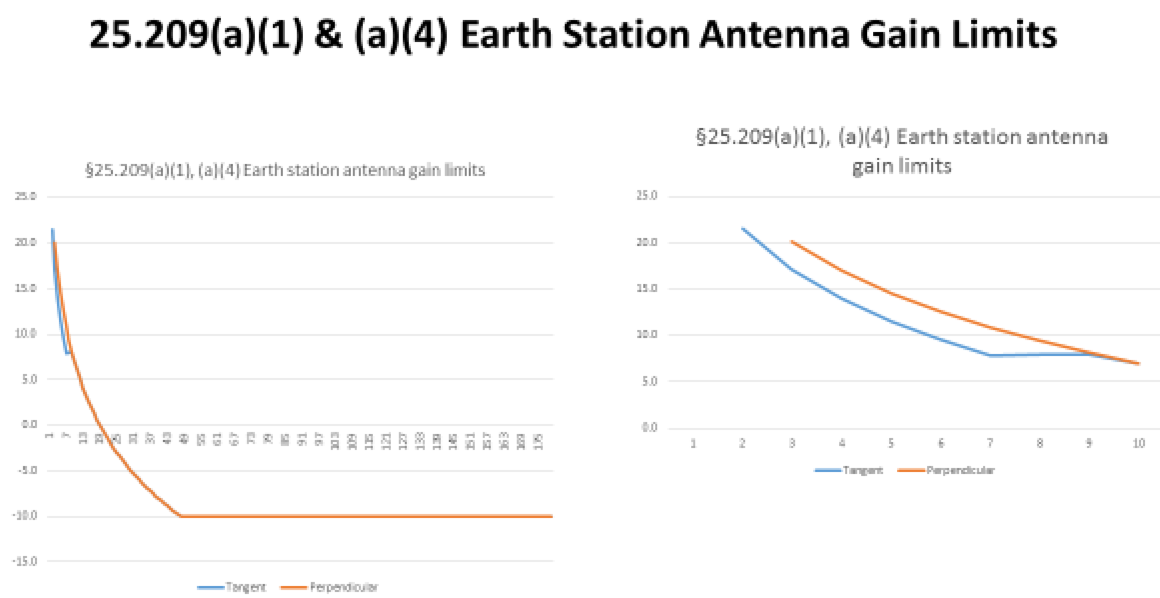

Figure 4 – 25.209(a)(1) & (a)(4) Earth Station Antenna Gain Limits

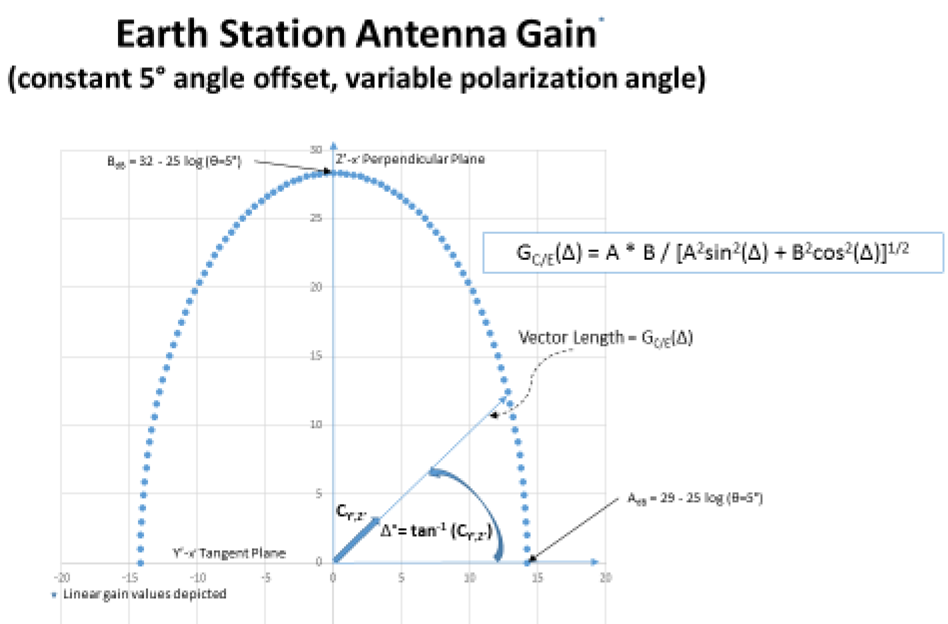

Figure 5 – Earth Station Antenna Gain

METHOD 2

This method uses a number of modified spherical coordinate systems to relate position and orientation of the three principals (earth station, CBSD, and satellite) to each other. The method uses vector projection of a relative position vector along the unit vectors of a modified spherical coordinate systems to determine relative spatial orientation of the principles. .

Modified Spherical Coordinate System

The spherical coordinate system can be particularly useful for positioning of points of interest on the surface of the earth. As shown below, with a slight modification to this coordinate system, the latitude and longitude of a point of interest can be used to positon the point on a sphere. On the other hand, rectangular coordinate system (henceforth denoted as Earth Centered Earth Fixed (ECEF)) is especially useful for calculating distances between points of interest. In this calculation we will require the use of both coordinate systems. The transformation from the ECEF to spherical coordinate system is given by the following equation (see [1, 3]):

$$\underline{\hat{r}} = sin\theta cos\varphi\ \underline{\hat{x}} + \ sin\theta sin\varphi\ \underline{\hat{y}} + cos\theta\ \underline{\hat{z}}$$

$$\underline{\hat{\theta}} = cos\theta cos\varphi\ \underline{\hat{x}} + \ cos\theta sin\varphi\ \underline{\hat{y}} - sin\theta\ \underline{\hat{z}}$$

$$\underline{\hat{\varphi}} = -sin\varphi \quad \underline{\hat{x}} + \quad cos\varphi\ \underline{\hat{y}} + \ 0 \quad \underline{\hat{z}}$$

These equations can be presented in the matrix form are as follows (see VII-12a in [1]):

$$\begin{bmatrix} \underline{\hat{r}} \\ \underline{\hat{\theta}} \\ \underline{\hat{\varphi}} \end{bmatrix} = \begin{bmatrix} sin\theta cos\varphi & sin\theta sin\varphi & cos\theta \\ cos\theta cos\varphi & cos\theta sin\varphi & -sin\theta \\ -sin\varphi & cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \underline{\hat{x}} \\ \underline{\hat{y}} \\ \underline{\hat{z}} \end{bmatrix}$$

In the spherical coordinate system the angle φ is defined in the counter clockwise fashion in the x-y plane with respect to x axis. In a similar way, the longitude λ is defined in the counter clockwise fashion in the equatorial plane with respect to prime meridian. However, the definition of θ differs from latitude *L* in that θ is defined with respect to z axis, while *L* is defined with respect to the equatorial plane (see Figure 6). Adjusting the spherical coordinate system for this difference results in the following matrix form of transformation from the ECEF system to modified spherical coordinate system:

$$\begin{bmatrix} \underline{\hat{r}} \\ \underline{\hat{\theta}} \\ \underline{\hat{\varphi}} \end{bmatrix} = \begin{bmatrix} cosLcos\lambda & cosLsin\lambda & sinL \\ sinLcos\lambda & sinLsin\lambda & -cosL \\ -sin\lambda & cos\lambda & 0 \end{bmatrix} \begin{bmatrix} \underline{\hat{x}} \\ \underline{\hat{y}} \\ \underline{\hat{z}} \end{bmatrix}$$

With respect to a particular point of reference on the surface of the earth, determination of relative position of other points of interest requires a local coordinate system. One such local coordinate system is the East, North, and Up (ENU), as shown in the Figure 6. The transformation from the ECEF system to ENU is given by[20]

[20] It should be noted that $\underline{\hat{N}}$ points in the negative direction of $\underline{\hat{\theta}}$, while $\underline{\hat{U}}$ and $\underline{\hat{E}}$ point in the same direction as $\underline{\hat{r}}$ and $\underline{\hat{\varphi}}$ respectively.

$$\begin{bmatrix} \hat{\underline{E}} \\ \hat{\underline{N}} \\ \hat{\underline{U}} \end{bmatrix} = \begin{bmatrix} -sin\lambda & cos\lambda & 0 \\ -sinLcos\lambda & -sinLsin\lambda & cosL \\ cosLcos\lambda & cosLsin\lambda & sinL \end{bmatrix} \begin{bmatrix} \hat{\underline{x}} \\ \hat{\underline{y}} \\ \hat{\underline{z}} \end{bmatrix} \quad (1)$$

Where L is latitude and λ is longitude of a point on the surface of the earth. The reverse transformation is given by

$$\begin{bmatrix} \hat{\underline{x}} \\ \hat{\underline{y}} \\ \hat{\underline{z}} \end{bmatrix} = \begin{bmatrix} -sin\lambda & -sinLcos\lambda & cosLcos\lambda \\ cos\lambda & -sinLsin\lambda & cosLsin\lambda \\ 0 & cosL & sinL \end{bmatrix} \begin{bmatrix} \hat{\underline{E}} \\ \hat{\underline{N}} \\ \hat{\underline{U}} \end{bmatrix} \quad (2)$$

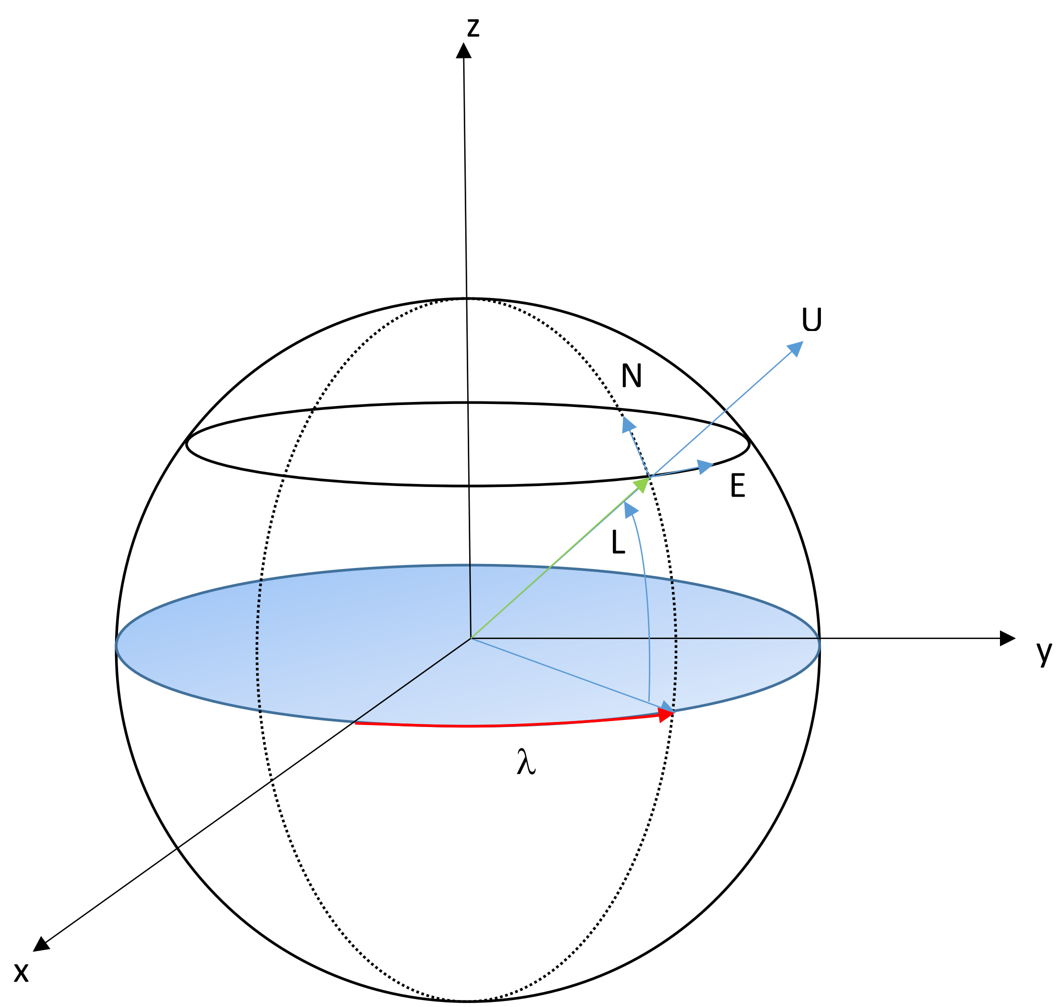

Figure 6 ENU local coordinate system in relation to ECEF coordinate system

Earth Station Local Coordinate System

The position of an earth station (ES) on the surface of the earth in the ECEF coordinate system can be expressed as[21]:

$$\underline{R}_{ECEF}^{ES} = R_e \left(cosL_{ES} cos\lambda_{ES} \, \underline{\hat{x}} + cosL_{ES} sin\lambda_{ES} \, \underline{\hat{y}} + \ sinL_{ES} \, \underline{\hat{z}} \right) \tag{2.5}$$

Where R_e is the radius of the earth. A similar expression exists for the position of the CBSD, $\underline{R}_{ECEF}^{CBSD}$. A vector pointing from the ES to CBSD can be expressed in terms of position vectors of ES and CBSD as

$$\underline{\rho}_{ECEF}^{ES-CBSD} = \ \underline{R}_{ECEF}^{CBSD} \text{ - } \underline{R}_{ECEF}^{ES} \tag{3}$$

Using this vector, the azimuth and elevation of the CBSD relative to ES can now be expressed in the local coordinate system as follows (See Figure 7):

$$AZ_{ENU}^{CBSD} = atan \left(\frac{\underline{\hat{\rho}}_{ECEF}^{ES-CBSD} \cdot \underline{\hat{E}}^{ES}}{\underline{\hat{\rho}}_{ECEF}^{ES-CBSD} \cdot \underline{\hat{N}}^{ES}} \right) \tag{4}$$

$$EL_{ENU}^{CBSD} = asin \left(\underline{\hat{\rho}}_{ECEF}^{ES-CBSD} \cdot \ \underline{\hat{U}}^{ES} \right) \tag{5}$$

Where

$$\underline{\hat{\rho}}_{ECEF}^{ES-CBSD} = \frac{\underline{\rho}_{ECEF}^{ES-CBSD}}{\left\| \underline{\rho}_{ECEF}^{ES-CBSD} \right\|} \tag{6}$$

[21] Equation (1) can be used to transform the ECEF coordinate of the ES into a local coordinate system.

And where $\underline{\hat{E}}^{ES}$, $\underline{\hat{N}}^{ES}$, and $\underline{\hat{U}}^{ES}$ are respectively the East, North, and Up unit vectors at the ES.

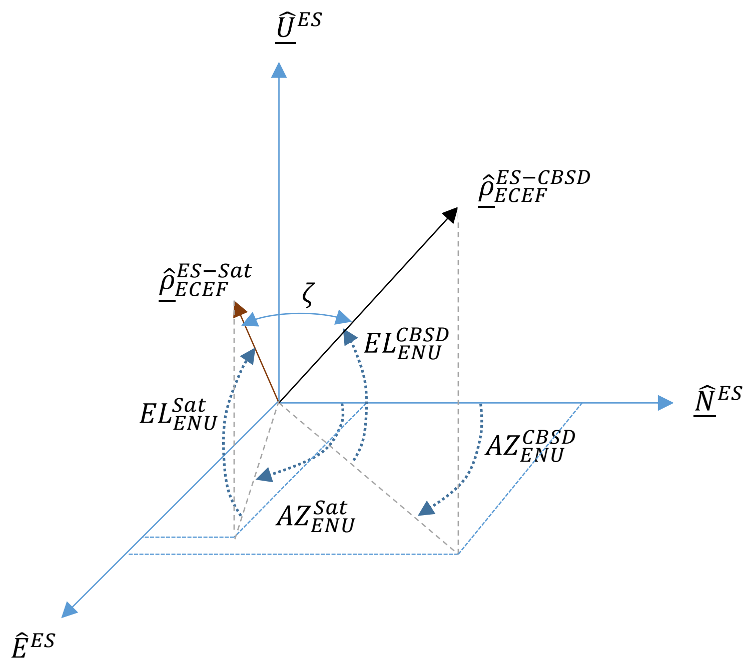

Figure 7 Azimuth and elevation in reference to ES-ENU

Similarly, the azimuth and elevation of the Satellite relative to ES can be expressed in the local coordinate system as

$$AZ_{ENU}^{Sat} = atan\left(\frac{\underline{\hat{\rho}}_{ECEF}^{ES-Sat} \cdot \underline{\hat{E}}^{ES}}{\underline{\hat{\rho}}_{ECEF}^{ES-Sat} \cdot \underline{\hat{N}}^{ES}}\right) \tag{7}$$

$$EL_{ENU}^{Sat} = asin\left(\underline{\hat{\rho}}_{ECEF}^{ES-Sat} .\ \underline{\hat{U}}^{ES}\right) \tag{8}$$

Satellite Local Coordinate System

To maximize the received power at the ES, the polarization angle of the ES antenna must be aligned with the polarization angle of the signal transmitted by the satellite. As such, a local coordinate system at the satellite is necessary to facilitate the alignment of the polarization angles. Figure 8 shows one such local coordinate system, with $\underline{\hat{U}}^{Sat}$ pointing away from the earth, and with $\underline{\hat{E}}^{Sat}$ and $\underline{\hat{N}}^{Sat}$ parallel and perpendicular to the equatorial plane respectively. For linearly polarized waves, the vertical polarization corresponds to $\underline{\hat{N}}^{Sat}$ and horizontal polarization corresponds to $\underline{\hat{E}}^{Sat}$. According to [2] “the polarization angle at the earth station is the angle between the plane defined by the local vertical at the earth station and the antenna boresight, and the polarization plane.” This relationship can be expressed by the following equation:

$$\psi = \text{atan}\left(\frac{\underline{\widehat{U}}^{ES} . \underline{\widehat{E}}^{Sat}}{\underline{\widehat{U}}^{ES} . \underline{\widehat{N}}^{Sat}}\right) \tag{9}$$

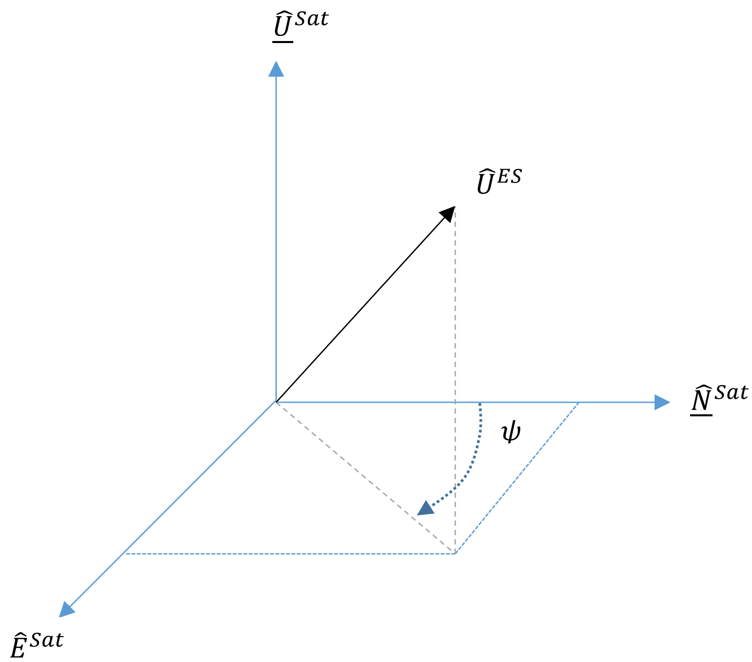

Figure 8 Projection of $\underline{\widehat{U}}^{ES}$ in Sat-ENU

It should be noted that the ES antenna must be rotated to compensate for the skew in the polarization angle ψ in the negative direction.

Earth Station Antenna Local Coordinate System

Figure 9 shows ES antenna local coordinate system, where $\underline{\hat{B}}^{Ant}$ is a unit vector in the direction of the ES antenna boresight, and $\underline{\hat{V}}'^{Ant}$ and $\underline{\hat{H}}'^{Ant}$ are unit vectors in the direction of the vertical and horizontal polarization of the ES antenna. This coordinate system[22] serves as an intermediate coordinate system for the tilted ES antenna coordinate system discussed below. The relationship between the H'V'B coordinate system and ENU coordinate system is similar to relationship between ENU and ECEF coordinate systems. Hence, $\underline{\hat{H}}'^{Ant}$, $\underline{\hat{V}}'^{Ant}$, and $\underline{\hat{B}}^{Ant}$ unit vectors can simply be written by inspection of equation (1) as follows:

$$\begin{bmatrix} \underline{\hat{H}}'^{Ant} \\ \underline{\hat{V}}'^{Ant} \\ \underline{\hat{B}}^{Ant} \end{bmatrix} = \begin{bmatrix} -sin(\xi_{ENU}^{Sat}) & cos(\xi_{ENU}^{Sat}) & 0 \\ -sin(EL_{ENU}^{Sat})cos(\xi_{ENU}^{Sat}) & -sin(EL_{ENU}^{Sat})sin(\xi_{ENU}^{Sat}) & cos(EL_{ENU}^{Sat}) \\ cos(EL_{ENU}^{Sat})cos(\xi_{ENU}^{Sat}) & cos(EL_{ENU}^{Sat})sin(\xi_{ENU}^{Sat}) & sin(EL_{ENU}^{Sat}) \end{bmatrix} \begin{bmatrix} \underline{\hat{E}}^{ES} \\ \underline{\hat{N}}^{ES} \\ \underline{\hat{U}}^{ES} \end{bmatrix} \tag{10}$$

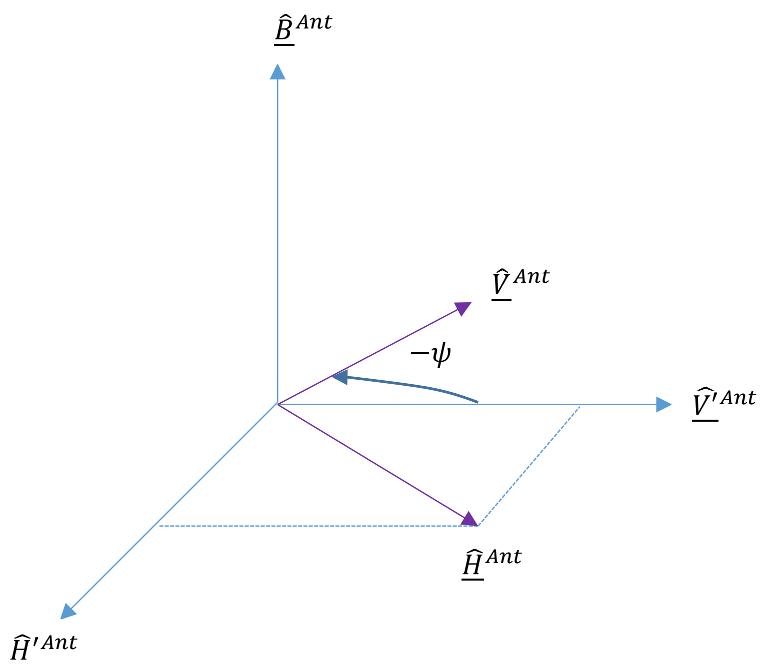

Figure 9 ES antenna H'V'B coordinate System

Where $\xi_{ENU}^{Sat} = \pi/2 - AZ_{ENU}^{Sat}$. The $(\pi/2 - AZ_{ENU}^{Sat})$ factor is to account for how the AZ_{ENU}^{Sat} is defined.

Tilted Earth Station Antenna Local Coordinate System

In order to account for the skew in the polarization angle, a local coordinate system for the tilted antenna is necessary to define the relationship of the ES with other potential sources of interference such as CBSD. Figure 9 shows the superposition of HVB coordinate system (the tilted coordinate system) on top of the H'V'B coordinate system. $\underline{\hat{V}}^{Ant}$ and $\underline{\hat{H}}^{Ant}$ are unit vectors in the direction of the tilted vertical and

[22] This local coordinate system also allows for an alternative, and perhaps more intuitive, way of calculating polarization angle at the ES.

horizontal polarization of the ES antenna. The relationship between the $\underline{\hat{V}}^{Ant}$ and $\underline{\hat{H}}'^{Ant}$ and $\underline{\hat{V}}'^{Ant}$ can be expressed as

$$\underline{\hat{V}}^{Ant} = sin(-\psi)\ \underline{\hat{H}}'^{Ant} + cos(-\psi)\ \underline{\hat{V}}'^{Ant}$$

Similarly,

$$\underline{\hat{H}}^{Ant} = cos(-\psi)\ \underline{\hat{H}}'^{Ant} - sin(-\psi)\ \underline{\hat{V}}'^{Ant}$$

These equations can be presented in the matrix form are as follows:

$$\begin{bmatrix} \underline{\hat{H}}^{Ant} \\ \underline{\hat{V}}^{Ant} \\ \underline{\hat{B}}^{Ant} \end{bmatrix} = \begin{bmatrix} cos(-\psi) & -sin(-\psi) & 0 \\ sin(-\psi) & cos(-\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{\hat{H}}'^{Ant} \\ \underline{\hat{V}}'^{Ant} \\ \underline{\hat{B}}^{Ant} \end{bmatrix} \tag{11}$$

Expressed in terms of ENU coordinate system

$$\begin{bmatrix} \underline{\hat{H}}^{Ant} \\ \underline{\hat{V}}^{Ant} \\ \underline{\hat{B}}^{Ant} \end{bmatrix} = \begin{bmatrix} cos(-\psi) & -sin(-\psi) & 0 \\ sin(-\psi) & cos(-\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\text{x} \begin{bmatrix} -sin(\xi_{ENU}^{Sat}) & cos(\xi_{ENU}^{Sat}) & 0 \\ -sin(EL_{ENU}^{Sat})cos(\xi_{ENU}^{Sat}) & -sin(EL_{ENU}^{Sat})sin(\xi_{ENU}^{Sat}) & cos(EL_{ENU}^{Sat}) \\ cos(EL_{ENU}^{Sat})cos(\xi_{ENU}^{Sat}) & cos(EL_{ENU}^{Sat})sin(\xi_{ENU}^{Sat}) & sin(EL_{ENU}^{Sat}) \end{bmatrix} \begin{bmatrix} \underline{\hat{E}}^{ES} \\ \underline{\hat{N}}^{ES} \\ \underline{\hat{U}}^{ES} \end{bmatrix} \tag{12}$$

Application of this coordinate system is discussed in the next section.

Earth Station Antenna Gain Performance

Section 25.209 specifies ES antenna performance along the plane tangent to the GSO arc (and in the plane perpendicular to the GSO arc) as function angle θ in degrees from a line from the ES antenna to the assigned orbital location of the target satellite. With respect to interference from a CBSD, the position of the CBSD will generally not be exactly in the GSO tangent plane or in the plane perpendicular to GSO arc. To account for this deviation, it is necessary to scale the antenna performance for positions of the CBSD in between these principle planes. One such scaling is the elliptic scaling[23] given by the following equation:

$$G(\eta) = \frac{a\ b}{\sqrt{(b\ cos\eta)^2 + (a\ sin\eta)^2}} \tag{13}$$

Where η is the angle that the unit vector from ES to CBDS in the ENU coordinate systems makes with $\underline{\hat{H}}^{Ant}$ in the $\underline{\hat{H}}^{Ant}\underline{\hat{V}}^{Ant}$ plane. Parameters *a* and *b* are respectively the antenna performance requirements along the GSO arc and perpendicular to GSO arc in linear scale, as specified in §25.209. Figure 10 shows the angle of the unit vector from CBDS to ES in the tilted ES antenna local coordinate system.

[23] The impetus for choosing such a scaling is the elliptical nature of the antenna that is specified in §25.209.

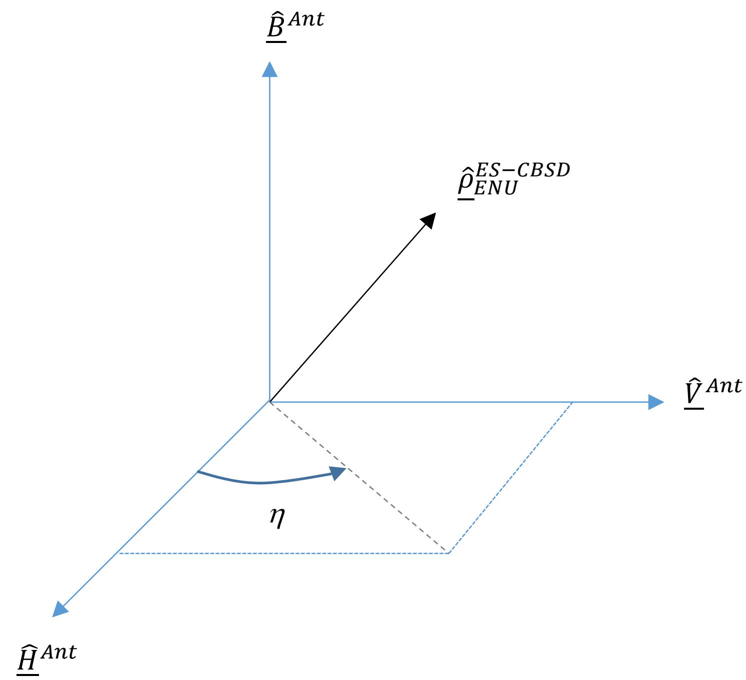

Figure 10 ES antenna HVB coordinate system

Where $\underline{\hat{\rho}}_{ENU}^{ES-CBSD}$ is a unit vector given by

$$\underline{\hat{\rho}}_{ENU}^{ES-CBSD} = \begin{bmatrix} -sin\lambda_{ECEF}^{ES} & cos\lambda_{ECEF}^{ES} & 0 \\ -sinL_{ECEF}^{ES}cos\lambda_{ECEF}^{ES} & -sinL_{ECEF}^{ES}sin\lambda_{ECEF}^{ES} & cosL_{ECEF}^{ES} \\ cosL_{ECEF}^{ES}cos\lambda_{ECEF}^{ES} & cosL_{ECEF}^{ES}sin\lambda_{ECEF}^{ES} & sinL_{ECEF}^{ES} \end{bmatrix} \underline{\hat{\rho}}_{ECEF}^{ES-CBSD} \quad (14)$$

As shown in Figure 10, the angle η can be determined by the following equation:

$$\eta = atan\left(\frac{\underline{\hat{\rho}}_{ENU}^{ES-CBSD} \cdot \underline{\hat{V}}^{Ant}}{\underline{\hat{\rho}}_{ENU}^{ES-CBSD} \cdot \underline{\hat{H}}^{Ant}}\right) \quad (15)$$

The angle θ, which is the angle between a unit vector from the earth station to CBSD ($\underline{\hat{\rho}}_{ECEF}^{ES-CBSD}$) and a unit vector from the earth station to the target satellite ($\underline{\hat{\rho}}_{ECEF}^{ES-Sat}$) is shown in Fig. 6 as ζ. This angle is given by

$$\zeta = acos\left(\underline{\hat{\rho}}_{ECEF}^{ES-CBSD} \cdot \underline{\hat{\rho}}_{ECEF}^{ES-Sat}\right) \quad (16)$$

Satellite Azimuth and Elevation from Look Angle

Many of the above equations depend on knowing the satellite azimuth (and elevation in case of NGSO) in ECEF coordinate system. It may be the case that this information may not be directly available and may have to be determined from satellite look angle in local coordinate system (ENU). Below, this relationship is derived.

The position of the Satellite in the ECEF is given by the following equation

$$\underline{R}_{ECEF}^{Sat} = R_{Sat}\,(cosL_{ECEF}^{Sat}cos\lambda_{ECEF}^{Sat}\ \underline{\hat{x}} + cosL_{ECEF}^{Sat}sin\lambda_{ECEF}^{Sat}\ \underline{\hat{y}} + \ sinL_{ECEF}^{Sat}\ \underline{\hat{z}}) \quad (17)$$

From this equation the satellite azimuth λ_{ECEF}^{Sat} and elevation[24] L_{ECEF}^{Sat} can be determined in the following manner

$$\lambda_{ECEF}^{Sat} = atan\left(\frac{\underline{\hat{R}}_{ECEF}^{Sat} \cdot \underline{\hat{y}}}{\underline{\hat{R}}_{ECEF}^{Sat} \cdot \underline{\hat{x}}}\right) \tag{18}$$

And

$$L_{ECEF}^{Sat} = asin\, \underline{\hat{R}}_{ECEF}^{Sat} \cdot \underline{\hat{z}} \tag{19}$$

where

$$\underline{\hat{R}}_{ECEF}^{ES} = \frac{\underline{R}_{ECEF}^{ES}}{\left\|\underline{R}_{ECEF}^{ES}\right\|} = \frac{\underline{R}_{ECEF}^{ES}}{R_{Sat}} \tag{20}$$

Vector $\underline{R}_{ECEF}^{Sat}$ can also be expressed in the ECEF coordinate system (see Figure 11) as

$$\underline{R}_{ECEF}^{Sat} = \underline{\rho}_{ECEF}^{ES-Sat} + \underline{R}_{ECEF}^{ES} = \underline{\hat{\rho}}_{ECEF}^{ES-Sat} \left\|\underline{\rho}_{ECEF}^{ES-Sat}\right\| + \underline{R}_{ECEF}^{ES} \tag{21}$$

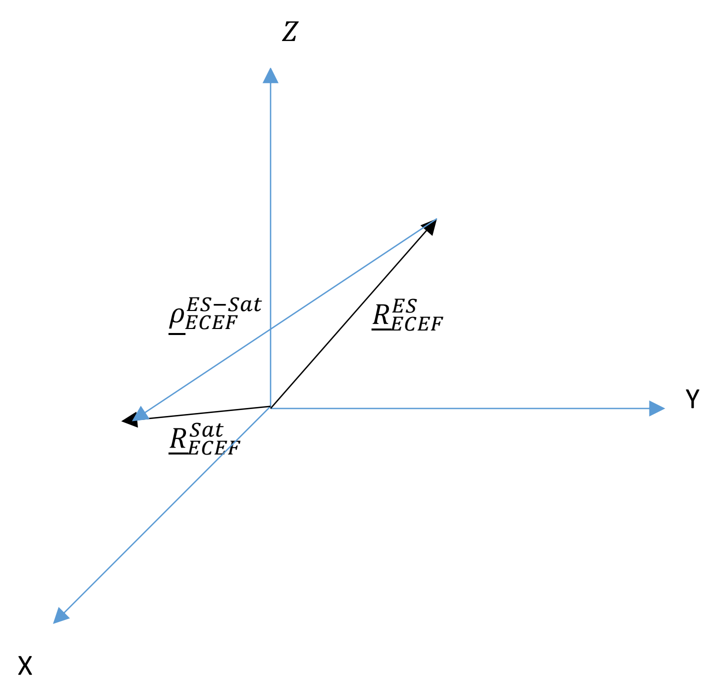

Figure 11 Earth station and satellite position vectors in reference to ECEF

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[24] For GSO Satellite elevation angle is zero.

As the location of the ES is known then $\underline{R}_{ECEF}^{ES}$ can be determined using equation (2.5). Vector $\underline{\rho}_{ECEF}^{ES-Sat}$ is given by inverse transform from ENU to ECEF of $\underline{\hat{\rho}}_{ENU}^{ES-Sat}$ (see equation (2))

$$\underline{\hat{\rho}}_{ECEF}^{ES-Sat} = \begin{bmatrix} -sin\lambda_{ECEF}^{ES} & -sinL_{ECEF}^{ES}cos\lambda_{ECEF}^{ES} & cosL_{ECEF}^{ES}cos\lambda_{ECEF}^{ES} \\ cos\lambda_{ECEF}^{ES} & -sinL_{ECEF}^{ES}sin\lambda_{ECEF}^{ES} & cosL_{ECEF}^{ES}sin\lambda_{ECEF}^{ES} \\ 0 & cosL_{ECEF}^{ES} & sinL_{ECEF}^{ES} \end{bmatrix} \underline{\hat{\rho}}_{ENU}^{ES-Sat} \tag{22}$$

Where

$$\underline{\hat{\rho}}_{ENU}^{ES-Sat} = \frac{\underline{\rho}_{ENU}^{ES-Sat}}{\left\| \underline{\rho}_{ENU}^{ES-Sat} \right\|} = \left(cosL_{ENU}^{Sat}cos\xi_{ENU}^{Sat}\ \underline{\hat{E}}^{ES} + cosL_{ENU}^{Sat}sin\xi_{ENU}^{Sat}\ \underline{\hat{N}}^{ES} + \ sinL_{ENU}^{Sat}\ \underline{\hat{U}}^{ES}\right) \tag{23}$$

Using law of cosines the following quadratic equation is obtained:

$$\left\| \underline{\rho}_{ECEF}^{ES-Sat} \right\|^2 - 2\left\| \underline{\rho}_{ECEF}^{ES-Sat} \right\| \left(\underline{\hat{\rho}}_{ECEF}^{ES-Sat} . \underline{R}_{ECEF}^{ES} \right) + \left({R_{ES}}^2 - {R_{Sat}}^2 \right) = 0 \tag{24}$$

Which has the following valid solution

$$\left\| \underline{\rho}_{ECEF}^{ES-Sat} \right\| = \left(\underline{\hat{\rho}}_{ECEF}^{ES-Sat} . \underline{R}_{ECEF}^{ES} \right) + \sqrt{\left(\underline{\hat{\rho}}_{ECEF}^{ES-Sat} . \underline{R}_{ECEF}^{ES} \right)^2 - \left({R_{ES}}^2 - {R_{Sat}}^2 \right)} \tag{25}$$

Substituting equation (25) into (21) provides the solution for $\underline{R}_{ECEF}^{Sat}$, from which satellite azimuth λ_{ECEF}^{Sat} and elevation L_{ECEF}^{Sat} are determined using equations (18) and (19) respectively.

METHOD 3

This method uses coordinate system rotations to determine the angular position of the GSO satellite and the terrestrial CBSD relative to the FSS earth station. This is followed by vector analysis using the local earth station antenna pattern coordinate system, to calculate the angle and gain of the earth station in the direction of the fixed terrestrial CBSD.

Let us define the location of the earth station, space station (satellite), and terrestrial station in Earth-Centered, Earth-Fixed (ECEF) Polar coordinates. We can then easily convert to ECEF Cartesian coordinates, which are more useful for later calculations. In the ECEF coordinate system, the origin is the center of the Earth. In polar coordinates, we can define the location of the stations in terms of ρ (rho), the radius of the Earth plus the elevation of the station's antenna;[25] λ (lambda), the latitude of the station; and φ (phi), the longitude of the station. In Cartesian coordinates, we can define the locations of the stations in terms of X, Y, and Z coordinates, where the origin of the coordinate system is the center of the Earth, the positive X axis passes through 0° latitude, 0° longitude, the positive Y axis passes through 0° latitude, 90° east longitude, and the positive Z axis passes through the geographic North Pole. Refer to Figure 12 below.

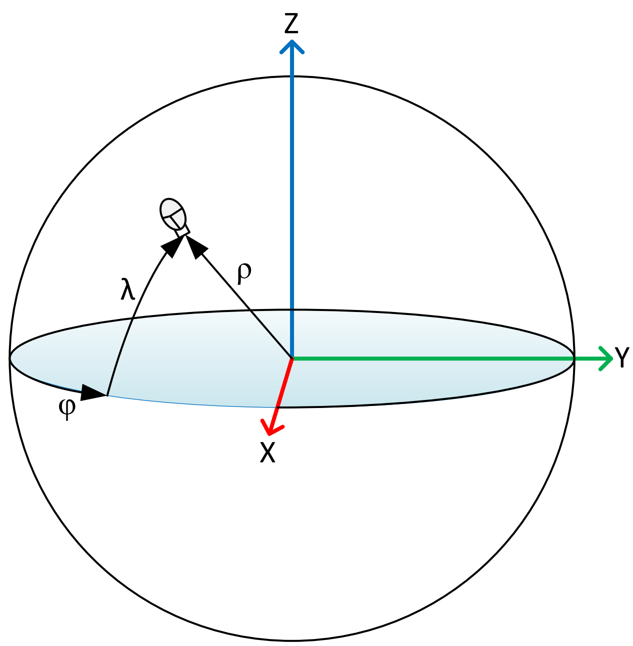

Figure 12: Earth-Centered, Earth-Fixed Coordinates of the Earth Station

To convert the earth station, satellite, and terrestrial station polar coordinates to Cartesian coordinates, we use the following equations, which are expressed in vector form, assuming a spherical model of the Earth:

[25] For geostationary satellites, we can assume the satellite is at an altitude of 35,786 kilometers above the Equator.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \rho\cos\lambda\cos\varphi \\ \rho\cos\lambda\sin\varphi \\ \rho\sin\lambda \end{bmatrix}$$

Knowing the location of the earth station in the ECEF coordinate system allows us to rotate the satellite and terrestrial station coordinates in the ECEF system to the Local Horizontal Plane (LHP) coordinate system, which is a coordinate system whose origin is the earth station location, and in which the X axis points to the Earth's north geographic pole, the Y axis points due east, and the Z axis points toward the center of the Earth.

Our end goal, however, is to express the coordinates of the terrestrial station in the Antenna Pattern coordinate system. The Antenna Pattern coordinate system is a right-handed Cartesian coordinate system, whose origin is the earth station location. In the Antenna Pattern coordinate system, the X axis points to the target satellite, the Y axis is in the plane tangent to the geostationary satellite arc at the location of the target satellite, as viewed from the earth station, and the Z axis completes the right-handed Cartesian coordinate system, which in this case means it points down. To accomplish that, we will first rotate the satellite and terrestrial station ECEF coordinates to the LHP coordinate system. Then we can determine the azimuth, elevation, and skew angles[26] of the target satellite. Knowing those angles will allow us to rotate the coordinate system axes from the LHP coordinate system to the Antenna Pattern coordinate system.

Coordinate system axis rotations in three dimensions can be accomplished by multiplying the X, Y, and Z coordinate vectors by 3 x 3 rotation matrices, very similar to those used to rotate vectors while maintaining the same coordinate system. The difference between the rotation matrices for axes and those for rotating vectors while keeping the same axes is that the signs of the sines in the X, Y, and Z axis rotation matrices are the opposite of those in the X, Y, and Z axis vector rotation matrices. The coordinate axis rotation matrices are as follows:

$$X_{rot}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$Y_{rot}(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$Z_{rot}(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[26] The azimuth angle of the satellite with respect to the earth station is the angle from the projection of the earth station to satellite vector projected onto the Local Horizontal Plane, measured clockwise (when viewed from above) from True North. The elevation angle is the angle from the earth station to satellite vector measured in the plane perpendicular to the Local Horizontal Plane. The skew angle is the angle between the local vertical plane at the earth station location and the plane perpendicular to the geostationary arc at the location of the target satellite, or the angle between the local horizontal plane at the location of the earth station and the plane tangent to the geostationary arc at the location of the target satellite as viewed from the earth station.

To rotate the ECEF coordinate axes to the LHP coordinate system, we first rotate around the Y axis by -90°. That rotates the positive X axis into the former positive Z axis location, and the positive Z axis rotates into the equatorial plane at 90° East Longitude. Refer to Figure 13 below:

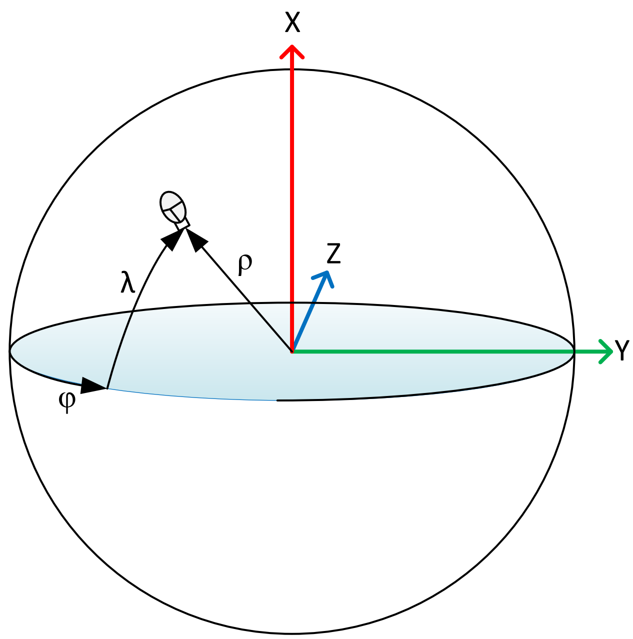

Figure 13: After rotation by -90 degrees around the Y axis

The next step is to rotate around the new X axis by the longitude of the earth station. Refer to Figure 14 below:

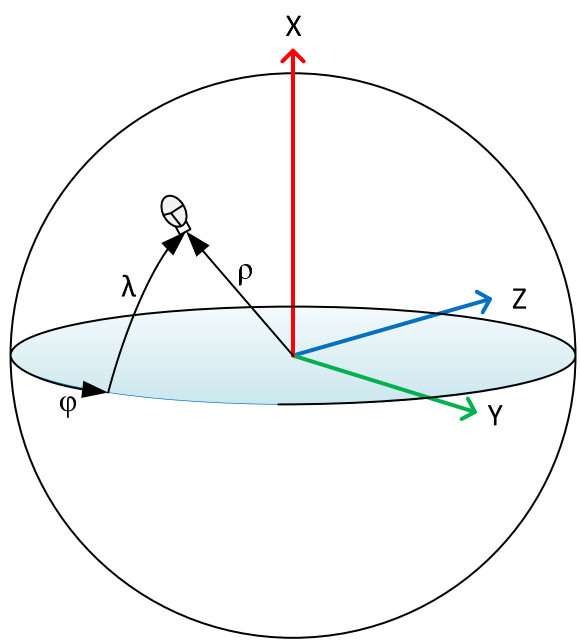

Figure 14: After rotation by the earth station longitude around the new X axis

The third step is to rotate around the new Y axis by the negative of the latitude of the earth station. Refer to Figure 15 below:

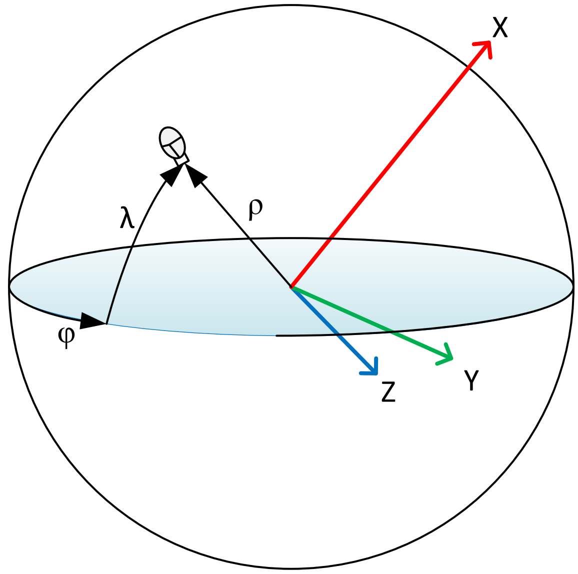

Figure 15: After rotation by the negative of the earth station latitude

The final step in converting the ECEF coordinate axes to LHP coordinate axes is to translate the origin of the coordinate system shown in Figure 15 along the new Z axis by ρ (rho). Refer to Figure 16 below:

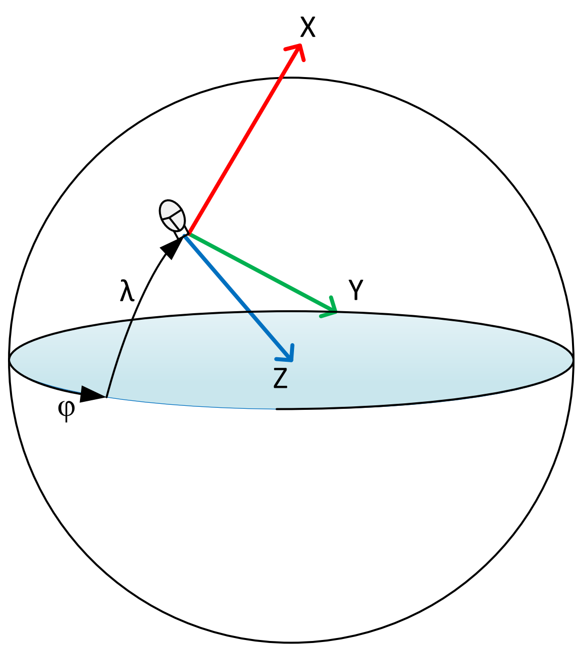

Figure 16: After translation along the Z axis by rho

The three rotations shown above can be performed by multiplying the station x, y, z coordinates by the following matrix:

$$ECEFtoLTP_{rot} = \begin{bmatrix} -\sin\lambda\cos\varphi & -\sin\lambda\sin\varphi & \cos\lambda \\ -\sin\varphi & \cos\varphi & 0 \\ -\cos\lambda\cos\varphi & -\cos\lambda\sin\varphi & -\sin\lambda \end{bmatrix}$$

Therefore the operation of transforming and translating from ECEF to LHP coordinates for any of the station vectors can be expressed as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{LTP} = ECEFtoLTP_{rot} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ECEF} + \begin{bmatrix} 0 \\ 0 \\ \rho_{ES} \end{bmatrix},$$

Where ρ_{ES} is the sum of the elevation of the earth station (distance above the radius of the Earth) and the radius of the Earth at the earth station's location.

Now that we have translated the ECEF coordinate axes into the Local Horizontal Plane coordinate system, let's look at that system in more detail. Refer to Figure 17 below:

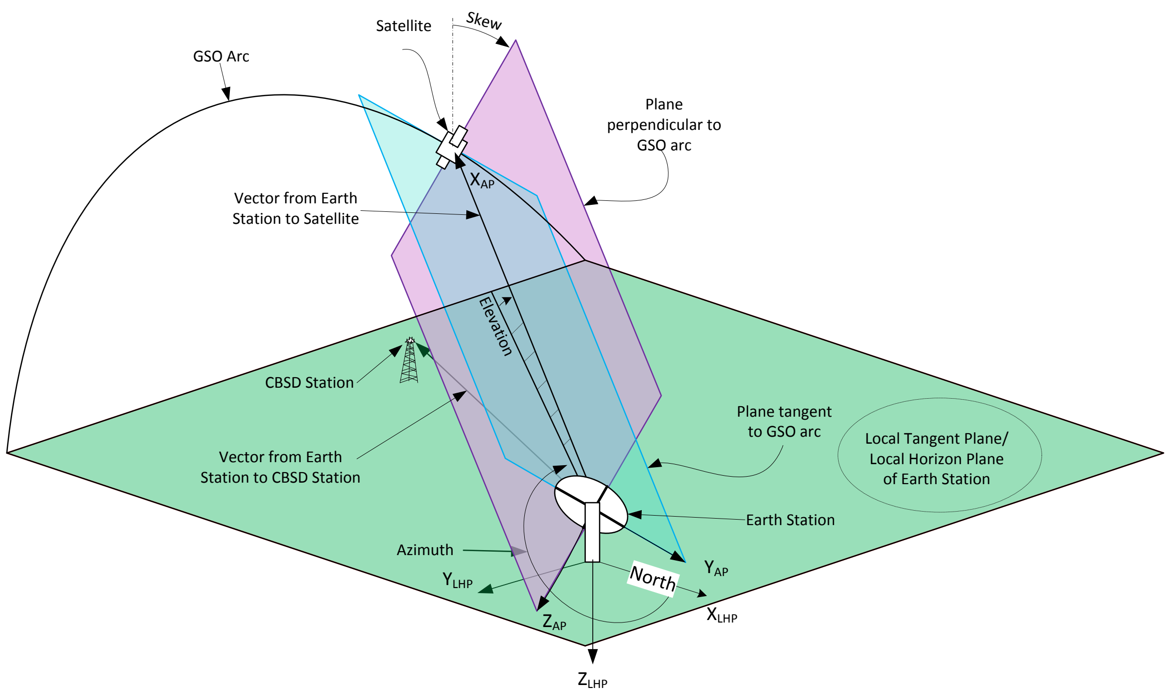

Figure 17: Local Horizontal Plane and Antenna Pattern Coordinate Systems

The azimuth angle of the satellite as viewed from the earth station, which we will denote as α, is the arctangent of the y component of the vector from the earth station to the satellite divided by the x component,[27] expressed as:

$$\alpha = \tan^{-1}\left(\frac{y_{EStoSat}}{x_{EStoSat}}\right)$$

The elevation angle of the satellite as viewed from the earth station, which we will denote as ϵ, is the arctangent of the –z component of the vector from the earth station to the satellite divided by the square root of the sum of the squares of the x and y components, expressed as:

$$\epsilon = \tan^{-1}\left(\frac{-z_{EStoSat}}{\sqrt{x_{EStoSat}{}^2 + y_{EStoSat}{}^2}}\right)$$

The skew angle can be calculated in several different ways. One way is to project the vector from the center of the Earth to the earth station, which is normal to the local horizontal plane at the location of the earth station, onto the plane perpendicular to the vector from the center of the Earth to the satellite, and then find the angle between the projected vector and the vertical (north-south) axis of the plane. Finding this angle will be facilitated by rotating the ECEF coordinate system around its Z axis by the longitude of the satellite.[28]

Let us denote the vector from the center of the Earth to the earth station as *ES*, and the vector from the center of the Earth to the satellite as *SAT*. We can project *ES* onto the plane perpendicular to *SAT* using the following vector equation:

$$V = |ES| - [|ES| \cdot |SAT|] \cdot |SAT|,$$

where $|ES|$ and $|SAT|$ are the normalized *ES* and *SAT* vectors. Then we can compute a vector *W* in which we rotate the coordinate axes around the Z axis as follows:

$$W = Z_{rot}(\varphi_{SAT}) \cdot V.$$

And finally, the skew angle, which we will denote as σ, is computed as the arctangent of the z component of *W* divided by the y component of W:

$$\sigma = \tan^{-1}\left({}^{W_Z}/_{W_Y}\right).$$

Once we have determined the azimuth, elevation, and skew angles that relate the Local Horizontal Plane coordinate system to the Antenna Pattern coordinate system, we can rotate the axes of the Local Horizontal Plane coordinate system into the Antenna Pattern coordinate system. Then we can compute the off-axis angle of the vector from the earth station to the terrestrial station with respect to the X axis of the Antenna Pattern coordinate system. We can also compute the rotation angle of the vector from the earth

[27] Because the azimuth of the satellite as viewed from the earth station ranges from 0 to 360 degrees, the use of a four-quadrant arctangent function is recommended.

[28] Because we are dealing with geostationary satellites, we can assume the latitude of the satellite is 0 degrees.

station to the terrestrial station with respect to the Y and Z axes of the Antenna Pattern coordinate system. These two angles and the applicable Section 25.209(a) antenna gain masks give us what we need to calculate the gain of the earth station antenna in the direction of the terrestrial station.

To rotate the Local Horizontal Plane coordinate system axes into the Antenna Pattern coordinate system, we perform the following computation on the terrestrial station coordinates in the Local Horizontal Plane coordinate system: First, we rotate around the Z axis of the Local Horizontal Plane coordinate system by the azimuth of the satellite α as seen from the earth station. Then we rotate around the new Y axis by the elevation angle of the satellite ε as seen from the earth station. Finally, we rotate around the new X axis by the skew angle σ. In equation form, the computation is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{CBSD\_AP} = X_{rot}(\sigma) \cdot Y_{rot}(\epsilon) \cdot Z_{rot}(\alpha) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{CBSD_{LTP}}.$$

The rotation matrix $X_{rot}(\sigma) \cdot Y_{rot}(\epsilon) \cdot Z_{rot}(\alpha)$ multiplies out to:

$$LTPtoAP_{rot} = \begin{bmatrix} \cos\alpha \cdot \cos\epsilon & \sin\alpha \cdot \cos\epsilon & -\sin\epsilon \\ \cos\alpha \cdot \sin\epsilon \cdot sin\sigma - \sin\alpha \cdot \cos\sigma & \cos\alpha \cdot \cos\sigma + \sin\alpha \cdot \sin\epsilon \cdot \sin\sigma & \cos\epsilon \cdot \sin\sigma \\ \sin\alpha \cdot \sin\sigma + \cos\alpha \cdot \sin\epsilon \cdot \cos\sigma & \sin\alpha \cdot \sin\epsilon \cdot \cos\sigma - \cos\alpha \cdot \sin\sigma & \cos\epsilon \cdot \cos\sigma \end{bmatrix}.$$

Then the offset angle of the terrestrial station with respect to the X axis of the Antenna Pattern coordinate system, which we will denote as θ_{offset} is computed as follows:

$$\theta_{\text{offset}} = \tan^{-1}\left[\sqrt{{y_{CBSD_{AP}}}^2 + {z_{CBSD_{AP}}}^2} \Big/ x_{CBSD_{AP}}\right].$$

And the rotation angle of the terrestrial station in the Y-Z plane of the Antenna Pattern coordinate system, which we will denote as θ_{rot} can be computed as follows[29]:

$$\theta_{\text{rot}} = \tan^{-1}\left[{z_{CBSD_{AP}}} / {y_{CBSD_{AP}}}\right].$$

For an example, and for ease of reference, tables in Section 25.209(a)(1) and (4) are reproduced below:

According to Section 25.209(a)(1), the gain of the earth station antenna in the plane tangent to the GSO arc, which corresponds to the X-Y plane of the Antenna Pattern coordinate system, shall be less than the following levels:

29-25log$_{10}\theta_{\text{offset}}$ dBi, for 1.5° ≤ θ_{offset} ≤ 7°

[29] A four-quadrant arctangent is recommended for the computation of this angle.

8 dBi, for 7° < θ_{offset} ≤ 9.2°
32-25log$_{10}$ θ_{offset} dBi, for 9.2° < θ_{offset} ≤ 48°
−10 dBi, for 48° < θ_{offset} ≤ 180°

Table 1: Section 25.209(a)(1), Gain Tangent to the GSO Arc

And in the plane perpendicular to the GSO arc, which corresponds to the X-Z plane of the Antenna Pattern coordinate system, Section 25.209(a)(4) prescribes that the earth station antenna gain shall be less than the following levels:

32-25log$_{10}$ θ_{offset} dBi, for 3° < θ_{offset} ≤ 48°
−10 dBi, for 48° < θ_{offset} ≤ 180°

Table 2: Section 25.209(a)(4), Gain Perpendicular to the GSO Arc

Comparing and contrasting these two tables, we see that the gain in the two planes is the same over the range of 9.2° to 180° off-axis. The gain is only defined in the X-Y plane of the Antenna Pattern coordinate system between 1.5° and 3° off-axis, but it is unlikely that a terrestrial station would be located that close to the vector from the earth station to the target satellite, because most satellites are at significantly higher elevation angles as seen from the earth station with which they are communicating.

The range of 3° to 9.2° off-axis is interesting, because the gain in the plane perpendicular to the GSO arc is twice as high (in power ratio terms, 3 dB higher in decibel terms) as the plane tangent to the GSO arc over the range of 3° to 7°, and the gain in the plane tangent to the GSO arc is fixed at 8 dBi over the range of 7° to 9.2°.

We can interpret these tables as meaning there are isogain contours centered on the vector from the earth station to the target satellite in the range of 3° to 7° off-axis. Because the gain in the plane perpendicular to the GSO arc is twice[30] that of the gain in the plane tangent to the GSO arc in that range, we see that the isogain contours will be elliptical in shape, with a major axis twice as long as the minor axis, and the major axis aligned with the plane perpendicular to the GSO arc. Thus, we can compute the gain of the earth station antenna, given Tables 1 and 2 above and the angles θ_{offset} and θ_{rot}, as follows. First, compute the gain in the plane tangent to the GSO arc (the X-Y plane of the Antenna Pattern coordinate system), which we will denote as G_{tan}, and the gain in the plane tangent to the GSO arc (the X-Z plane of the Antenna Pattern coordinate system), which we will denote G_{perp}:

$$G_{tan} = 10^{\max(2.9-2.5\log_{10}(\theta_{\text{offset}}),0.8)}, \text{ and}$$

$$G_{perp} = 10^{(3.2-2.5\log_{10}(\theta_{\text{offset}}))}.$$

Let us denote the earth station antenna gain as G_{es}. If $\theta_{\text{offset}} < 3°$, the terrestrial station is in or near the main lobe of the antenna, so set G_{es} to 45 dBi.

If $3° \leq \theta_{\text{offset}} < 9.2°$, calculate G_{es} using the polar equation of an ellipse,[31] as follows:

[30] Twice the gain in ratio terms, or 3 dB higher in decibel terms.
[31] Lawrence, J. Dennis. "Section 3.4: Ellipse." *A Catalog of Special Plane Curves*. New York: Dover Publications, 1972. Print.

$$G_{es} = 10\log\left[\frac{G_{tan}\cdot G_{perp}}{\sqrt{G_{perp}^2\cdot\cos^2\theta_{rot} + G_{tan}^2\cdot\sin^2\theta_{rot}}}\right].$$

If $9.2° \le \theta_{\text{offset}} < 48°$, calculate G_{es} as follows:

$$G_{es} = 10\log(G_{perp}).$$

If $48° \le \theta_{\text{offset}} < 180°$, G_{es} = -10 dBi.

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REFERENCES

[1] Constantine A. Balanis, “Antenna Theory”, Third Edition, Wiley, 2005, pgs. 1083-1084

[2] Gerard Maral and Michel Bousquet, “Satellite Communications Systems”, 5th Edition, Wiley, 2009, Sec. 8.3.5

[3] William H. Hayt, “Engineering Electromagnetics”, McGraw-Hill, 1981, pgs. 21-25

[4] Michael Geyer, “Aircraft Navigation and Surveillance Analysis for a Spherical Earth”, DOT-VNTSC-FAA-15-01, Federal Aviation Administration; October 2014, Section 4.2, ‘The Indirect Problem of Geodesy’

https://ntl.bts.gov/lib/53000/53100/53123/DOT-VNTSC-FAA-15-01.pdf